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Optimal Rate for Constant-Fidelity Entanglement in Quantum Communication Networks

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Abstract

In this paper, we propose an entanglement scheme for long-distance, constant-fidelity communication in quantum networks. We discuss the optimal rate of entanglement that allows for constant fidelity in both elementary and multihop links. We also discuss time complexity and propose the mathematical order of the rate capacity for an entanglement scheme. We propose a recursive entanglement scheme, a simultaneous entanglement scheme, and an adjacent entanglement scheme mathematically analyze these schemes. The rate capacity of the recursive and simultaneous entanglement schemes is $\Omega(1/e^n)$, but the adjacent entanglement scheme performs better, providing a rate of $\Omega(1/n)$.

Keywords

rate capacity; constant fidelity; entanglement scheme; ad hoc quantum networks

1 Introduction

Quantum processing and communication networks combine the disciplines of quantum mechanics and classical information science. However, quantum networks are more secure and efficient than classical networks [1]-[3]. Quantum communication networks are based on quantum teleportation, which is essential for transmitting quantum states and establishing quantum channels between nodes at a distance [2]. Quantum teleportation exploits one of the most intriguing properties of quantum mechanics: entanglement. In quantum entanglement, Einstein, Podolsky, Rosen (EPR) pairs are generated and then distributed via fibers or optical free-space links. This allows different nodes to transport a quantum state by transmitting classical bits rather than quantum bits [2].

At present, photons are primarily used as the carrier in quantum entanglement generation. They are used over distances of some hundreds of kilometers (e.g 250 km on fiber links and 144 km on free space) [3]. However, the success of entanglement decreases exponentially with distance because of absorption loss and detector noise in the transmission channel.

A solution to entanglement delay is to insert quantum repeaters between different nodes [5], [6]. Quantum repeaters can store quantum states and use protocols to perform quantum operations. A repeater protocol defines a sequence of three operations: entanglement generation on elementary links, entanglement swapping, and entanglement purification [7]. The protocol should be well designed to maximize success at each link.

Entangled photons are used to express quantum states as EPR pairs. With the development of quantum networks, quantum optics has attracted more research interests recently. Fidelity is one of the most important topics in this field and is defined as the probability that a decoded message has the same amount of information after coding and transmission as it did before coding and transmission. High fidelity is essential in long-distance communication to make messages reliable. In long-distance entanglement generation, fidelity can be affected by memory decay, local measurement errors, entanglement swapping, or entanglement purification failure [7]. The upper bound of the entanglement generation rate between nodes connected by quantum repeaters is determined by these factors.

The quantum entanglement rate of each elementary link can be calculated and measured for specific quantum ad hoc networks. In a multihop network, optimized quantum repeater protocols increase the rate of quantum entanglement to reach the upper bound.

There are two main approaches to optimizing quantum repeater protocols: developing a new quantum repeater structure (using quantum mechanics), and adapting classical network concepts to the quantum network. In [8], the former approach is taken. A Duan, Lukin, Cirac, Zoller (DLCZ) protocol generates and connects entangled pairs of atomic ensembles over short distances. The protocol couples a single photon with collective atomic excitation modes. In [9], Qubus (hybrid) repeaters are described. These repeaters perform operations on both local qubits and continuous-variable states (qubus). In [6], EPR generation rate in ad hoc quantum networks is investigated. Several heuristic algorithms are also introduced to ensure constant, maximum fidelity. These algorithms include Entanglement Swapping Scheme Search, and Shortest Path Entanglement Flow. Novel quantum broadcasting schemes, such as ring, angular, and regional broadcasting, are defined in a quantum repeater architecture and are extensions of classical broadcasting concepts [6]. Nevertheless, only recursive entanglement is applied in the quantum repeater protocols, and the entanglement rate when the number of nodes tends to infinity has not yet been determined.

In this paper, we focus on the rate capacity of entanglement that ensures constant fidelity when the number of nodes in a multihop link tends towards infinity. We use concepts from classical ad hoc networks to optimize the rate in a quantum repeater with a typical structure [7]. We propose three entanglement schemes for quantum repeater protocols. These schemes are recursive entanglement, simultaneous entanglement, and adjacent entanglement. Through mathematical analysis, we show that recursive entanglement and simultaneous entanglement provide a rate on the order of $1/e^n$. However, adjacent entanglement provides a rate on the order of $1/n$, which indicates better performance.

In section 2, we provide background on the entanglement rate capacity for an elementary link. In section 3, we describe the entanglement rate capacity model of EPR pairs for a multihop link. In section 4, we propose three entanglement schemes that ensure constant fidelity to establish multi-hop links are defined, the mathematical derivation of the rate capacity for each scheme is presented, and their performances are analyzed.

2 Entanglement Rate Capacity Model for an Elementary link

For elementary link, the average time τ_{ij} to generate an EPR pair is given as the sum of the detection time $1/fP_g$ and communication time d_{ij}/c:

$$\tau_{ij}(f, d, P_g) = (1/fP_g) + (d_{ij}/c) \tag{1}$$

where d_{ij} is the Euclidian distance between two consecutive nodes i and j; f is the operating frequency in the quantum repeater; P_g is the probability of realizing an EPR pair; and c is the speed of light. For the quantum repeater in [7], P_g is

$$P_g(F) = 0.5\left[1-(2F-1)^{\frac{2\eta}{1-\eta}}\right] \tag{2}$$

where F is the fidelity of the EPR pair between nodes, and η is the loss parameter, given by $e^{-\xi d}$, where d is the distance and ξ is the rate of loss. The operating frequency depends on the quantum memory space M and link distance d [7]:

$$M = 4\lceil df/c\rceil \tag{3}$$

Therefore, the maximum operating frequency is [7]

$$f_{\max} = Mc/4d \tag{4}$$

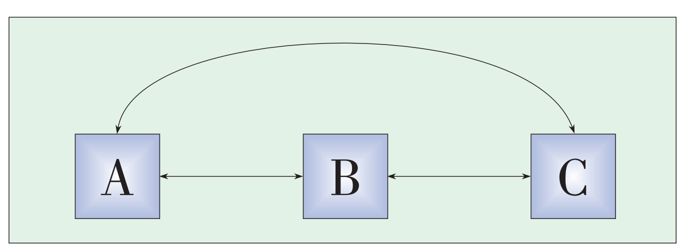
Communication time is much shorter than detection time, i.e. $(d/c) \ll (1/(fP_g))$. Therefore, the average EPR generation time is approximately equal to the detection time. A elementary link's capacity for generating EPR pairs can be defined as the reciprocal of the maximum average time to generate an EPR pair, and is given by

$$R(F) \equiv f_{\max}P_g(F) \tag{5}$$

3 Entanglement Rate Capacity Model for a Multihop Link

In practice, it is very difficult to transmit entangled photons over long distances because of channel loss and detector noise. It is reasonable to assume that entangled photons generated by a locate node can only be delivered to the consecutive node in a multihop link. A relay scheme is therefore used to ensure that any nodes over a long distance can share EPR pairs. Suppose Alice, Bob and Candy are three consecutive nodes in multihop link. Alice can transmit entangled photons directly to Bob but not to Candy. First, photons are entangled so that Alice can share the them with Bob through elementary link (A, B). At the same time, Bob shares entangled photos with Candy through elementary link (B,C). As a result, Bob has a photon from the EPR pair with Alice. He also has a photon from the EPR pair with Candy. Second, we perform entanglement connection on the two photons that Bob has. This involves consecutively swapping and purifying the entanglements. Third, we inform Alice and Candy about the classic information produced by the quantum operation. Alice and Candy use this information to operate the other entangled photons left in the quantum memory. Hence, by consuming two EPR pairs, we obtain an EPR pair between Alice and Candy, both of whom are unable to directly transmit photons. Using this relay scheme, more nodes can be added in multihop scenario, and fidelity can be constant for a remote node if entanglement swapping and purification are balanced.

In the case of three nodes (**Fig. 1**), we assume that entanglement occurs in the two elementary links at the same time. The average time for this process to occur has a lower bound of $\max\left\{\frac{1}{f_{AB}P_{g,AB}(F)}, \frac{1}{f_{BC}P_{g,BC}(F)}\right\}$. The probability of entanglement



▲**Figure 1. Entanglement for three nodes.**

connection is determined by the physical structure of the quantum repeater and is denoted P_c. The minimum average time τ_{AC} to obtain an EPR pair between Alice and Candy can be expressed as:

$$\tau_{AC} = \max\left\{\frac{1}{f_{AB}P_{g,AB}(F)}, \frac{1}{f_{BC}P_{g,BC}(F)}\right\}\frac{1}{P_c} \tag{6}$$

Moreover, the maximum average rate $R(A,C)$ can be expressed as

$$\begin{aligned} R(A,C) &= 1/\tau_{AC} \\ &= \min\{R(A,B), R(B,C)\}P_c \end{aligned} \tag{7}$$

For n consecutive hops, entanglement connection can be done on the kth repeater, and

$$R(0,N) = \min\{R(0,k), R(k,N)\}P_c \tag{8}$$

4 Three Entanglement Schemes for Establishing Multihop Links

To create an EPR pair between the source node and destination node, intermediate nodes are essential because they generate EPR pairs and follow swapping steps to transmit an entanglement. Take, for example, a finite multihop link with n nodes. If we know the exact entanglement rate between each pair of consecutive nodes, we can choose the appropriate sequence scheme for entanglement on a node. In this way, the rate of entanglement can be optimized. In this paper, we are mainly concerned with the upper bound on the rate capacity with high probability (w.h.p); i.e. with probability tending to 1 as n approaches infinity. In the following, we use a probabilistic variation to denote the mathematical order. As n approaches infinity, $f(n) = \Omega(g(n))$ w.h.p. as n approaches infinity. If there is a constant K, then $\lim_{n\to\infty} P(f(n) \geqslant Kg(n)) = 1$.

4.1 Recursive Entanglement Scheme

Recursive entanglement is an immediate scheme in which only one intermediate node is chosen for entanglement connection at every turn. This process is repeated recursively so that an EPR can be established between the source node and destination node in a multihop scenario. The start node can be selected arbitrarily, and entanglement connection should be done at recursively before or after the node.

If the source node is chosen as the start node, the process for an $n-1$ hop path in the ad hoc quantum network is as in **Fig. 2**. The arrows in Fig. 2 indicate EPR pairs between nodes, and the triangles below the nodes indicate entanglement connections. The recursive scheme involves $n-1$ steps:

1) Each node generates entanglements and shares an EPR pair with the adjacent nodes (Fig. 1). Node 1 shares an EPR pair with node 2, and node 2 shares an EPR pair with node 3. Node $n-1$ shares an EPR pair with node n.
2) Entanglement connection is done in node 2, and then node 2 sends the classical information to node 3 so that node 1 can share an EPR pair with node 3.
3) Entanglement connection is done in node 3, and then node 3 sends the classical information to node 4 so that node 1 can share an EPR pair with node 4.

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$n-1$) Entanglement connection is done in node $n-1$, and then node $n-1$ sends the classical information to node n so that node 1 can share an EPR pair with node n.

The time taken to send classical information is negligible. Therefore, by applying (8) recursively, we can conclude that

$$\begin{aligned} R(1,n) &= \min\{\min\{\min\{R(1,2), R(2,3)\}P_c, R(3,4)\}P_c, R(4,5)\}P_c \ldots \\ &= \min\{R(1,2)P_c^{n-2}, R(2,3)P_c^{n-3}, R(3,4)P_c^{n-4}, \ldots, R(n-1,n)\}P_c\} \end{aligned} \tag{9}$$

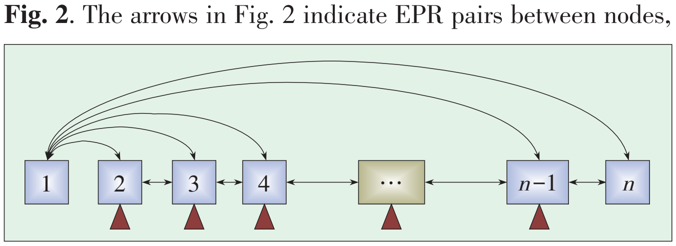
It is not necessary to choose node 1 as the starting node for the whole process if we know the exact entanglement generation rate of each elementary link. Then, we can use heuristic algorithms to determine which node to use as the start node and how to recursively do execute the process. In this way, the maximum value of $R(1,n)$ can be obtained. However, here we mainly focus on the mathematical order of $R(1,n)$. Considering there are always $n-1$ steps, we let $R(k\text{th})$ be the rate of the elementary link between the kth node and the next node at which entanglement is done. Then, we conclude that

$$\begin{aligned} R(1,n) &= \min\{R(1\text{st})P_c, R(2\text{nd})P_c^2, R(3\text{rd})P_c^3, \ldots, R((n-1)\text{th})P_c^{n-2}\} \\ &= r/a^{n-2} \\ &= \Omega(1/e^n) \text{ w.h.p. as } n \text{ approaches infinity} \end{aligned} \tag{10}$$

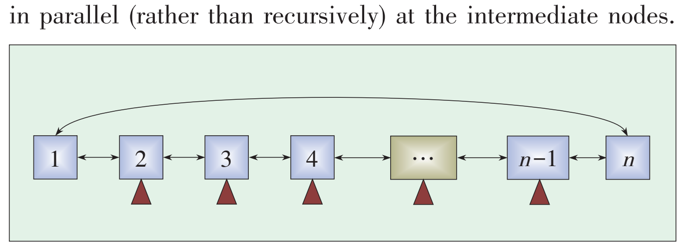
where r is a constant, and $a = 1/P_c$. The recursive entanglement scheme only provides a rate on the order of $1/e^n$.

4.2 Simultaneous Entanglement Scheme

Simultaneous entanglement was first described in [5]. A quantum routing mechanism was proposed to construct the quantum communication network. Here, we describe the mathematical order of the simultaneous entanglement swapping scheme (**Fig. 3**). This scheme allows entanglement connection in parallel (rather than recursively) at the intermediate nodes.



▲**Figure 2. Recursive entanglement when the source node is the start node.**



▲**Figure 3. Simultaneous entanglement.**

All the classical information is collected and applied to a specially designed quantum logical circuit in the destination node. Then, the source and destination are entangled. This scheme involves three steps:

1) Each node generates entanglements and shares an EPR pair with the adjacent nodes (Fig. 2). Node 1 shares an EPR pair with node 2, and node 2 shares an EPR pair with node 3. Node $n-1$ shares an EPR pair with node n.
2) Entanglement connection is done in parallel at all the intermediate nodes.
3) All the intermediate nodes send the classical information to node n. This information is received and processed at node n with a special quantum logical circuit. Node n then shares an EPR pair with node 1.

The time needed to send classical information is negligible, and all the $n-2$ entanglement connections should occur successfully at the same time. By applying (8) simultaneously, we conclude that

$$\begin{aligned} R(1,n) &= \min\{R(1,2), R(2,3), R(3,4), \ldots, R(n-1,n)\} P_c^{n-2} \\ &= r/a^{n-2} \\ &= \Omega(1/e^n) \text{ w.h.p. as } n \text{ approaches infinity} \end{aligned} \tag{11}$$

The simultaneous entanglement scheme only provides a rate on the order of $1/e^n$.

4.3 Adjacent Entanglement Scheme

In both the recursive and simultaneous entanglement schemes, the whole entanglement progress is a sequential chain of successful entanglement connections of each intermediate node. Therefore for an $n-1$ hop link the total probability is always presented as a constant multiplied by an $n-2$ power of P_c. To decrease the mathematical order of the entanglement rate, we propose using the adjacent entanglement scheme. The main idea of adjacent entanglement is to divide the multihop link into sets that contain adjacent nodes. This ensures that entanglement connection is done independent to that of other sets. After nodes in each set have performed entanglement connection, cluster sets are formed by small sets and continue with entanglement connection independently. Because entanglement connection occurs independent of other sets, the total probability is multiplied by P_c only once at every turn, and the order decreases. We describe a concise scenario in which there are only 2^k nodes in a multihop link (**Fig. 4**).

The adjacent entanglement scheme involves $2(\log_2 2^{k-1})$ steps:

1) Each node generates entanglements and shares an EPR pair with adjacent nodes (Fig. 4) Node 1 shares an EPR pair with node 2, and node 2 shares an EPR pair with node 3. Node $n-1$ shares an EPR pair with node n.
2) The link is divided into 2^{k-2} sets. Set 1 contains nodes 1 to 4, set 2 contains nodes 5 to 8 etc. Set i contains nodes $4i-3$ to $4i$, where i is an integer. Set 2^{k-2} contains nodes 2^k-3 to 2^k.
3) Entanglement connection is done at nodes 2, 3, 6, 7, ..., $4i-2$, $4i-1$ at the same time. These connections are independent. By the end of step 3, node 1 is entangled with node 4, node 5 is entangled with node 8 etc. Node $4i-3$ is entangled with node $4i$ etc.
4) Entanglement connection is done at nodes 4 and 5 so that node 1 is entangled with node 8 etc. Entanglement connection is done at nodes $8i-4$ and $8i-3$ etc.
5) Entanglement connection is done at nodes 8 and 9 so that node 1 is entangled with node 16 etc. Entanglement connection is done at nodes $16i-8$ and $16i-7$ etc.
6) Entanglement connection is done at nodes 16 and 17 so that node 1 is entangled with node 32 etc. Entanglement connection is done at node $32i-16$ and $32i-15$ etc.

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$2(\log_2 2^k-1)$. Entanglement connection is done at node 2^{k-1} and $2^{k-1}+1$ so that node 1 is entangled with node 2^k.

The time needed to send classical information is negligible, and each set is independent. Each time entanglement connection is done, P_c is only multiplied once because of this independence. Therefore, for a link with 2^k hops, the total probability is a constant multiplied by a $2(\log_2 2^k-1)$ power of P_c.

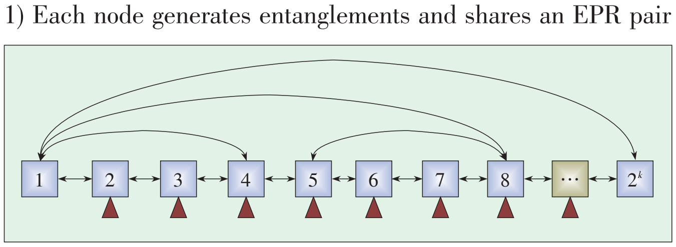
In a simple 8-node scenario,

$$\begin{aligned} R(1,8) = \min\{&\min[R(1,2), R(2,3), R(3,4)]P_c^2, R(4,5), \\ &\min[R(5,6), R(6,7), R(7,8)]\, P_c^2\}\, P_c^2 \end{aligned} \tag{12}$$

Therefore for an 8-hop link, the total probability is a constant multiplied by a $2(\log_2 8-1)$ power of successful connection probability P_c. For $n \to \infty$, the integer $k=\lceil \log_2 n \rceil$ makes $n \leqslant 2^k$, and

$$\begin{aligned} R(1,n) &\leqslant R(1,2^k) \\ \mathrm{R}(1,2^k) &= \Omega\left(\frac{r}{a^{2(\log_2 2^k-1)}}\right) \\ &= \Omega(1/a^k) \\ &\leqslant \Omega(1/n) \end{aligned} \tag{13}$$

Therefore, the adjacent entanglement scheme can only provide a rate on the order of $1/n$.



▲Figure 4. Adjacent entanglement scheme for an *a* node link.

5 Conclusion

In this paper, we have discussed the optimal rate of entanglement that can be achieved while still maintaining constant fidelity in a quantum communication network. Our discussion focused on a network in which the number of nodes in a multi-hop link approaches infinity. We proposed a recursive entanglement scheme, a simultaneous entanglement scheme, and an adjacent entanglement scheme. These ensure constant fidelity when establishing multihop links. However, mathematical analysis shows that the recursive and simultaneous entanglement schemes only provide a rate on the order of $1/e^n$ whereas the adjacent entanglement scheme performs better, providing a rate on the order of $1/n$.

References

[1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, "Quantum Entanglement," *Rev. Mod .Phys.*, vol. 81, no. 2, pp. 865-942, 2009.
[2] Malaney and A.Robert, "Location-dependent communications using quantum entanglement," *Phys. Rev.* A, vol. 81, no. 4, pp. 042319, 2010.
[3] K. Inoue, "Quantum key distribution technologies," *IEEE Journal of Quantum Electronics*, vol. 12, no. 4, pp. 888-896, July-Aug 2006.
[4] D. Stucki, "High rate, long-distance quantum key distribution over 250 km of ultra low loss fibres," *New J. Phys.*, vol. 11, 2009.
[5] S. T. Cheng, C. Y. Wang, and M. H. Tao, "Quantum communication for wireless wide-area networks," *IEEE JASC*, vol. 23, no. 7, pp. 1424-1432, July 2005.
[6] T. Bacinoglu, B. Gulbahar,and O.B. Akan, "Constant fidelity entanglement flow in quantum communication networks," in *Proc. IEEE GLOBECOM*, San Francisco, USA, 2010.
[7] B. He, Y.-H. Ren, and J. A. Bergou, "Creation of high-quality long-distance entanglement with flexible resources," *Phys. Rev.* A, vol. 79, no.5, PP. 0523232009.
[8] L. M. Duan, M. D. Lukin, J. I. Cirac, and P. Zoller, *Nature (London)*, pp 413-413, 2001.
[9] L. Jiang et al. "Quantum repeater with encoding, " *Phys. Rev. A*, vol. 79, no. 3, pp. 032325, 2009.

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